MATHEMATICS

INTUITIONIST TREATMENT OF SOME SPACES OF SEQUENCES

BY

SAHAB LAL SHUKLA

(Communicated by Prof. A. HEYTING at the meeting of September 25, 1971)

1. INTRODUCTION

In this article we propose to study certain sequence spaces in intuitionist functional analysis satisfying a condition of convergence or absolute convergence, or an analogous condition, and also some related spaces. This work forms a direct continuation of the results of [11], § 5. We refer the reader to [11], § 1 for a definition of all relevant notions in intuitionist functional analysis: at that place references are also given to the original articles in which the concerned definitions were introduced. We summarize here the points of departure from the conventions laid down there and establish some further notation which we shall use.

In the first place, the correct definition of quasi-number, given in [10], ought to be substituted for that cited in [11]. Secondly, we should more precisely say that a space B is *α-reflexive* if to every full linear functional Φ on B^* there corresponds an f in B such that $\Phi(f^*)=f^*(f)$ for every f^* in B^*, while also $|||\Phi|||=|||f|||$. The definitions of β-, γ- and δ-reflexivity should also be similarly interpreted. Thirdly, it may be remarked here that there is an apparant discrepancy between the statements of results in this article and in [11], where the notation of [18] has been used, the isomorphic equality in the context of dual spaces signifying a conjugate linear (anti-linear) one-one correspondence. In the present article, we use the term isomorphic equality to signify a linear one-one quasi-norm preserving correspondence (cf. e.g., [17]). Accordingly, whereas duality in [11] is studied in terms of the sesquilinear form (f, g), here we have rather used the bilinear form $[f, g]$ (which are both defined below).

The elements of our sequence spaces will be denoted by $f, g, h, \ldots$: in some of the spaces the sequences will be supposed to extend from the zero-th term ad infinitum, while in others, they will extend from the first term ad infinitum; the latter being more usual, the former will be made explicit whenever it occurs. The n-th component of a vector f will be denoted by $f(n)$, i.e.,

$$f=(f(1), f(2), f(3), \ldots)=(f(n))_{n=1}^{\infty},$$

or $$f=(f(0), f(1), f(2), \ldots)=(f(n))_{n=0}^{\infty}.$$

If the sequence $f(1), f(2), \ldots$ is of bounded variation then the quasi-number containing it will also be denoted by $f(\infty)$. The notation Δf will be used to denote the sequence $(\Delta f(n))_n$ where $\Delta f(n)=f(n)-f(n-1)$. In accordance with customary usage, $f(0), f(-1), f(-2), \ldots$ will be taken to be zero for a sequence $(f(n))_{n=1}^{\infty}$, while $f(-1), f(-2), f(-3), \ldots$ will be taken to be zero for a sequence $(f(n))_{n=0}^{\infty}$. For any f, ${}^nf$ will denote the vector given by ${}^nf(k)=f(k)$ for $k \leqslant n$ and ${}^nf(k)=0$ for $k>n$.

e_m denotes the vector given by $e_m(n)=\delta_{mn}$ where δ_{mn} is the Kronecker delta. e_0 is given by $e_0(0)=1$; $e_0(n)=0$ for $n=1, 2, \ldots$. e is the vector given by $e(n)=1$ for each n, i.e., $e=(1, 1, 1, \ldots)$.

For two sequences f and g, we shall use the notation $[f, g]$ to denote the infinite series $\sum f(n)\ g(n)$, and (f, g) to denote $\sum f(n)\ \overline{g(n)}$.

A particular distance delimitation, which will be used for the closed unit spheres of several different sequence spaces is ω given by the following specification: n lies in $\omega(f, 0)$ if $|f(k)|<2^{-n}$ for $1 \leqslant k \leqslant n$, $\omega(f, g)=\omega(f-g, 0)$. It follows immediately that ${}^nf$ ω-converges to f.

In the course of construction of various dual spaces we shall utilize the spread Λ whose direction consists of countably many nodes of order

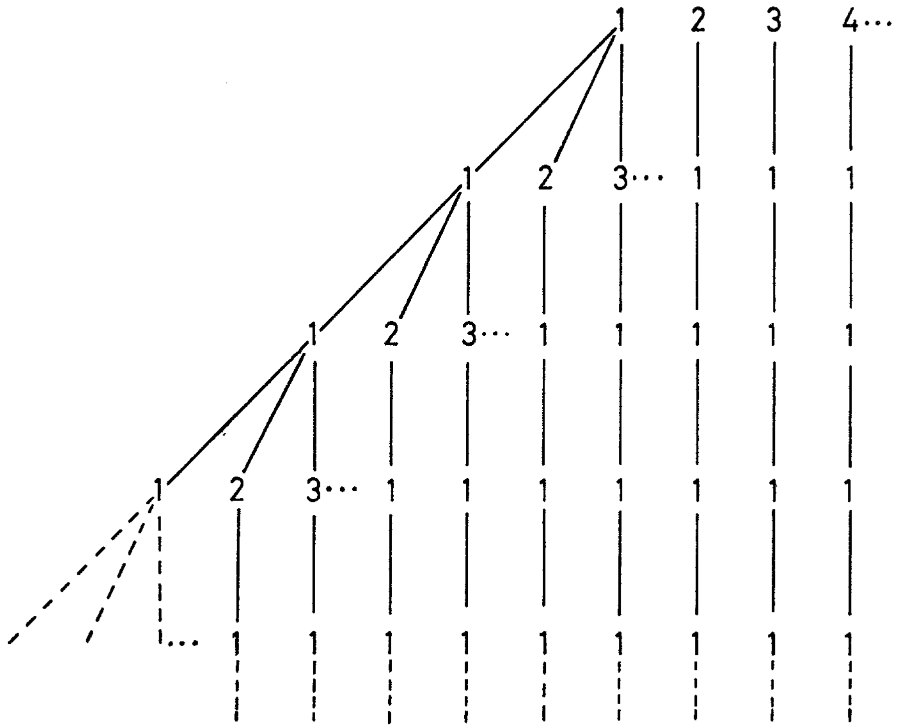

FIG. 1. THE SPREAD Λ

one, in which each node containing only 1's has countably many immediate descendants, while each node in which a natural number $n (n>1)$ occurs somewhere has exactly one immediate descendant. For technical ease of description later on, we dress the spread Λ as follows. The node (1^n) (which consists of n 1's) is dressed by n zeros, while a node $(1^m\ k\ 1^n)$ (which consists of m 1's followed by a k followed by n 1's) is dressed by m zeros followed by complex numbers $\eta_{m+1}, \eta_{m+2}, \ldots, \eta_{m+n+1}$ if $n \leqslant k-2$, while it is dressed by $\eta_{m+1}, \ldots, \eta_{m+k-1}$ followed by $n-k+2$ zeros if $n>k-2$.

The η_k's used will depend upon the space B is question, and, if necessary, this dependence will be indicated by writing $\eta_k(B)$. The corresponding dressed spread will be denoted $\Lambda(B)$.

The space bv consisting of all complex sequences of bounded variation (i.e., for which $\sum_{i=1}^{\infty} |\Delta f(i)|$ is bounded) equipped with the quasi-norm $|||f|||$ containing the bounded monotone sequence $t_2, t_3. \ldots$ where $t_n = \sum_{i=2}^{n} |\Delta f(i)| + |f(n)|$, and the normed linear space w^1 consisting of all f with convergent $\sum_1^{\infty} f(i)$ (the norm being $\|f\| = \sup_n (|\sum_{i=1}^{n} f(i)|)$ have been defined in [11], § 5 and it has been proved there that $(bv)^* = (bv)' = w^1$, $(w^1)^* = bv$ (loc. cit., theos. X, XI.)

2. The normed dual of w^1

First of all we determine the normed dual space of w^1. This space has not been determined in [11]. Let Φ be a full normed linear functional

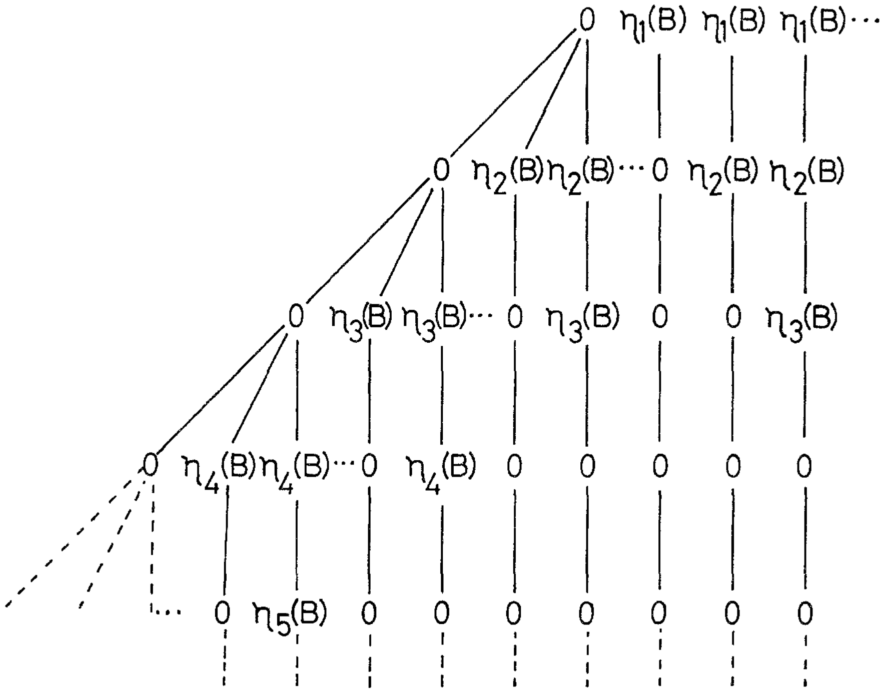

FIG.2. THE DRESSED SPREAD Λ(B)

defined on w^1. The vector g given by $g(k)=\Phi(e_k)$ lies in bv and $\sum_{i=2}^n |\Delta g(i)|+ |g(n)| \ngtr \|\Phi\|$ for every n, as shown in [11], theo. XI.

Since $\|\Phi\|$ is the supremum of Φ, we conclude that for each k, a vector f_k belonging to the closed unit sphere of w^1 can be indicated such that $\Phi(f_k)=[f_k, g]>\|\Phi\|-2^{-k-1}$, and now we can also indicate an $N=N(k)$ such that $|[{}^nf_k, g]-[f_k, g]|<2^{-k-1}$ for $n\geqslant N(k)$. Thus we have $\Phi({}^nf_k)=[{}^nf_k, g]> >[f_k, g]-2^{-k-1}>\|\Phi\|-2^{-k}$. However,

$$|\Phi({}^nf_k)|=|\sum_{i=1}^{n} f_k(i)g(i)|=|g(n)(\sum_{i=1}^{n} f_k(i))+(-\Delta g(n))\sum_{i=1}^{n-1} f_k(i)+\ldots+ +(-\Delta g(2))\, f_k(1)|$$

$$\ngtr |g(n)|\,|\sum_{i=1}^{n} f_k(i)|+|\Delta g(n)|\,|\sum_{i=1}^{n-1} f_k(i)|+\ldots+|\Delta g(2)|\,|f(1)|$$

$$\ngtr \sup_{1\leqslant i\leqslant n} |\sum_{j=1}^{i} f(j)|(|g(n)|+\sum_{i=2}^{n}|\Delta g(i)|) \ngtr 1\cdot(|g(n)|+\sum_{i=2}^{n}|\Delta g(i)|),$$

so that we have $\|\Phi\|-2^{-k}<(|g(n)|+\sum_{i=2}^n|\Delta g(i)|)\ngtr \|\Phi\|$, from which it follows that $\lim_{n\to\infty}(|g(n)|+\sum_{i=2}^{\infty}|\Delta g(i)|)$ exists and is equal to $\|\Phi\|$, i.e., $\|\Phi\|=\|g\|$.

We now introduce the space ac which is the species of all complex sequences f for which $\sum_{i=1}^{\infty}|\Delta f(i)|$ converges. The norm of any vector f is $\|f\|=\lim_{n\to\infty}(\sum_{i=2}^n|\Delta f(i)|+|f(n)|)$. Thus we have proved that to each full normed linear functional Φ on w^1 a vector g of ac can be assigned such that $\Phi(f)=[f, g]$.

Conversely, if g lies in ac, then it also lies in bv and determines a full linear functional Φ on w^1. We now prove that Φ possesses a supremum $\|g\|$ on the closed unit sphere of w^1. For each fixed k, for each n, we can choose exactly one assertion from those recognized as valid in each of the two sets of assertions: (i) $|\Delta g(n)|<2^{-k-n}$ and (ii) $|\Delta g(n)|>2^{-k-n-1}$ and (1) $|g(n)|<2^{-k-n}$ and (2) $|g(n)|>2^{-k-n-1}$. Now take $y_g^k(n)=0$ if (i) has been chosen and $y_g^k(n)=\overline{\operatorname{sgn}(-\Delta g(n+1))}$ if (ii) has been chosen; and $z_g^k(n)=0$ if (1) has been chosen and $z_g^k(n)=\overline{\operatorname{sgn} g(n)}$ if (2) has been chosen. Let $f_{n,k}=(\Delta y_g^k(1), \Delta y_g^k(2), \ldots, \Delta y_g^k(n-1), z_g^k(n)-y_g^k(n-1), 0, 0, 0, \ldots)$ $(y_g^k(0)=0)$ which belongs to w^1, so that

$$\Phi(f_{n,k})=\sum_{i=1}^{n-1} g(i)\,\Delta y_g^k(i)+g(n)\,(z_g^k(n)-y_g^k(n-1)).$$

However, $\sum_2^n|\Delta g(i)|+|g(n)|-2^{-k-2}-2^{-k-3}-\ldots-2^{-k-n+1}-2^{-k-n}-2^{-k-n}$ $\ngtr \sum_1^{n-1} y_g^k(i)\,(-\Delta g(i+1))+g(n)\,z_g^k(n)\ngtr \sum_1^{n-1} g(i)\,\Delta y_g^k(i)+g(n)\,(z_g^k(n)-y_g^k(n-1))\ngtr \Phi(f_{n,k})$ for every n. Now we determine an N so large that $\lim_{n\to\infty}\{\sum_2^n|\Delta g(i)|+|g(n)|\}-\{\sum_2^N|\Delta g(i)|+|g(N)|\}<2^{-k-1}$, so that $\Phi(f_{N,k})\nless\sum_2^N|\Delta g(i)|+|g(N)|-2^{-k-1}>\|g\|-2^{-k}$, from which follows that Φ is a normed linear functional on w^1. We state this result as

THEOREM I. $(w^1)'=ac$.

3. The space ac_0

ac_0 is the species of all those vectors f for which $\sum_{i=1}^{\infty} |\Delta f(i)|$ converges and $f(i)$ tends to zero as $i \to \infty$. The norm is $\|f\| = \lim_{n\to\infty} (\sum_{i=2}^{n} |\Delta f(i)| + f(n))$. The cataloguing of the closed unit sphere of ac_0 with respect to its norm is of the second kind.

Now we wish to determine the dual space of ac_0. Let Φ be a full linear functional defined on ac_0 and let $\Phi(e_k) = g(k)$, so that $\Phi(f) = \sum_{i=1}^{\infty} f(i)\, g(i)$ for each f lying in ac_0 in view of the linearity and the continuity of Φ implied by its fullness. Let $C \not< 0$ be any bound of Φ on the closed unit sphere of ac_0. Moreover, we have $|\Phi(\sum_{i=1}^{n} e_i)| = |\sum_{i=1}^{n} g(i)| \not> C$ for every n, since $\sum_{i=1}^{n} e_i$ lies in the closed unit sphere of ac_0. We now introduce the space d^1 which is the species of all complex sequences f for which $(|\sum_{i=1}^{n} f(i)|)_{n=1}^{\infty}$ is bounded. The quasi-norm of f is the quasi-number core containing the bounded monotone sequence

$$(|f(1)|, \max(|f(1)|, |\sum_{i=1}^{2} f(i)|), \max(|f(1)|, |\sum_{i=1}^{2} f(i)|, |\sum_{i=1}^{3} f(i)|, \ldots).$$

Conversely, if f belongs to the closed unit sphere of ac_0 and g lies in d^1, then $\sum_{i=1}^{\infty} f(i)g(i)$ converges as

$$\sum_{i=n}^{n+m} f(i)g(i) = \sum_{i=n}^{n+m} f(i)(\Delta G(i)) \text{ (where } G(i) = \sum_{j=1}^{i} g(j))$$
$$= \sum_{i=n}^{n+m-1} G(i)(-\Delta f(i+1)) + G(n+m)\, f(n+m) - f(n)\, G(n-1),$$

therefore,

$$|\sum_{i=n}^{n+m} f(i)g(i)| \not> \sum_{i=n}^{n+m-1} |\Delta f(i+1)|\, |G(i)| + |f(n)|\, |G(n-1)| +$$
$$+ |G(n+m)|\, |f(n+m)| \not>$$
$$\not> (\sum_{i=n}^{n+m-1} |\Delta f(i)| + |f(n)| + |f(n+m)|) \sup_{n-1 \leqslant i \leqslant n+m} |G(i)|$$
$$\not> (\sum_{i=n+1}^{n+m} |\Delta f(i)| + |f(n)| + |f(n+m)|) \cdot C$$

(where C is any bound of $|||g|||$ in d^1), which tends to zero as $n \to \infty$, since $\sum_{i=1}^{\infty} |\Delta g(i)|$ is convergent and $f(n)$ tends to zero as $n \to \infty$, so that $\Phi(f) = [f, g]$ defines a full linear functional on ac_0 and $|||\Phi||| = |||g|||$. We state our result as

Theorem II. $(ac_0)^* = d^1$.

It can be easily proved from the above discussion that $(ac_0)' = s^1$ where s^1 is the species of all complex sequences f for which $\sup_n |\sum_{i=1}^{n} f(i)|$ exists. It can also be proved in the same way as for the space l^{∞} which consists of all bounded sequences possessing suprema that s^1 is a non-linear space.

A one-one correspondence between s^1 and l^∞ is established by making the vector F of l^∞ corresponds to f of s^1 where $F(n) = \sum_1^n f(i)$. The reasoning used for proving $(l^\infty)^* = (l^\infty)' = l^1$ (cf. [12]) can now be adapted to prove that $(s^1)^* = (s^1)' = ac_0$, keeping in mind the fact that $(d^1)^* = (d^1)' = ac_0$, which is proved in the next section.

4. The space d^1

The closed unit sphere of d^1 permits a cataloguing of the first kind with respect to ω which follows in the same way as for b^p and b^∞ (cf. [11], § 2). We now proceed to determine the dual space of d^1 which can be obtained by considering the cataloguing of the closed unit sphere of d^1. We insert here an alternative proof without using the cataloguing.

Let Φ be a full linear functional defined on d^1 and let $g(k) = \Phi(e_k)$. For each fixed j, for each n, we can choose exactly one assertion each from those recognized as valid of the two sets of assertions: (i) $|\Delta g(n)| < 2^{-j-n}$ and (ii) $|\Delta g(n)| > 2^{-j-n-1}$; and (1) $|g(n)| < 2^{-j-n}$ and (2) $|g(n)| > 2^{-j-n-1}$. Now take $y_g{}^j(n) = 0$ if (i) has been chosen and $y_g{}^j(n) = \operatorname{sgn} \overline{(-\Delta g(n+1))}$ if (ii) has been chosen; and $z_g{}^j(n) = 0$ if (1) has been chosen and $z_g{}^j(n) = \operatorname{sgn} \overline{g(n)}$ if (2) has been chosen. Now we replace

$$(0, 0, 0, \ldots, 0, \eta_n, \eta_{n+1}, \eta_{n+2}, \ldots, \eta_{n+m}, 0, 0, \ldots)$$

(in the beginning $n-1$ zeros with $n \geqslant 1$) by

$$(0, 0, 0, \ldots, 0, y_g{}^j(n), \Delta y_g{}^j(n+1), \ldots, \Delta y_g{}^j(n+m-1), z_g{}^j(n+m) - \\ - y_g{}^j(n+m-1), 0, 0, \ldots)$$

to obtain the dressed spread $\Lambda(d^1)$. Since Φ is full on $\Lambda(d^1)$, it follows that to each k, an $N(k)$ can be assigned such that $|\Phi(0) - \Phi(f)| < 2^{-k}$ for any f in $\Lambda(d^1)$ passing through the dressed node consisting of N zeros. Applying this to the vector

$$(0, 0, 0, \ldots, 0, y_g{}^j(n), \Delta y_g{}^j(n+1), \Delta y_g{}^j(n+2), \ldots, \Delta y_g{}^j(n+m-1), \\ z_g{}^j(n+m) - y_g{}^j(n+m-1), 0, 0, \ldots)$$

(with initial $n-1$ zeros) for $n > N(k)$, we have

$$|g(n) y_g{}^j(n) + \sum_{i=n+1}^{n+m-1} g(i) \Delta y_g{}^j(i) + g(n+m)(z_g{}^j(n+m) - y_g{}^j(n+m-1))| < 2^{-k}.$$

However,

$$\left(\sum_{i=n+1}^{n+m} |\Delta g(i)| + |g(n+m)|\right) - 2^{-j-n-1} - \ldots - 2^{-j-(n+m)} - 2^{-j-(n+m+1)} \not> \\ \not> \left| \sum_{i=n}^{n+m-1} y_g{}^j(i)(-\Delta g(i+1)) + z_g{}^j(n+m) g(n+m) \right| \not> |g(n) y_g{}^j(n) + \\ + \sum_{i=n+1}^{n+m-1} g(i) \Delta y_g{}^j(i) + g(n+m)(z_g{}^j(n+m) - y_g{}^j(n+m-1)| < 2^{-k},$$

so that

$$\left(\sum_{i=n+1}^{n+m} |\Delta g(i)|+|g(n+m)|\right) \not> 2^{-k}+2^{-j-n}.$$

From this follows that $\sum_{i=1}^{\infty} |\Delta g(i)| (g(0)=0)$ is convergent, i.e., g lies in ac. However, the closed unit sphere of d^1 contains the closed unit sphere of b^1, so Φ is also full on b^1, which implies that g lies in c_0. Thus we have proved that to the full linear functional Φ of d^1, a vector g of ac_0 can be assigned such that $\Phi(f)=[f, g]$.

We now prove the continuity of Φ without using the cataloguing of the closed unit sphere of d^1. Given k, we first of all indicate an $N=N(k)$ such that $n \cdot 2^{-n} \cdot \|g\| \not> 2^{-k}$ for $n \geqslant N(k)$. If now f is in the closed unit sphere of d^1 and $n \geqslant N(k)$ lies in $\omega(f, 0)$ then $|\Phi(f)|=|\sum_{i=1}^{\infty} f(i)g(i)| \not> 2^{-k}$. For, taking $n \geqslant N(k)$, we have

$$\left|\sum_{i=1}^{n} f(i)g(i)\right|=\left|\sum_{i=1}^{n} g(i)\Delta F(i)\right| \text{ (where } F(i)=\sum_{j=1}^{i} f(j) \text{ and } F(0)=0)$$
$$=\left|\sum_{i=1}^{n-1} F(i)(-\Delta g(i+1))+g(n)F(n)\right| \not> \left(\sum_{i=2}^{n} |\Delta g(i)|+|g(n)|\right) \sup_{1 \leqslant i \leqslant n} |F(i)|$$
$$\not> \left(\sum_{i=2}^{n} |\Delta g(i)+|g(n)|\right) \sum_{i=1}^{n} |f(i)| \not> n \cdot 2^{-n}\left(\sum_{i=2}^{n} |\Delta g(i)|+\right.$$
$$\left.+|g(n)|\right) \not> n \cdot 2 \cdot^{-n} \|g\| \not> 2^{-k}.$$

We shall now prove that the full linear functional Φ has the supremum $\|g\|$. For, it is obvious that Φ is bounded by $\|g\|$ for all vectors f belonging to the closed unit sphere of d^1. Now taking the vector

$$f_{j,n}=(y_g{}^j(1), \Delta y_g{}^j(2), \ldots, \Delta y_g{}^j(n-1), z_g{}^j(n)-y_g{}^j(n-1), 0, 0, 0, \ldots)$$

for $n \geqslant N(k)$, we have

$$\Phi(f_{j,n})=g(1)\, y_g{}^j(1)+\sum_{i=2}^{n-1} g(i)\, \Delta y_g{}^j(i)+g(n)\,(z_g{}^j(n)-y_g{}^j(n-1))$$
$$=\sum_{i=1}^{n-1} y_g{}^j(i)(-\Delta g(i+1))+g(n)\, z_g{}^j(n) \not< \sum_{i=2}^{n} |\Delta g(i)|+|g(n)|-$$
$$-2^{-j-2}-2^{-j-3} \ldots -2^{-j-n+1}-2^{-j-n}-2^{-j-n}$$

i.e., $(\sum_{i=2}^{n} |\Delta g(i)|+|g(n)|)-2^{-j-1} \not> \Phi(f_{j,n})$, which proves that Φ possesses the supremum $\|g\|$.

Let f belong to the closed unit sphere of d^1 and g be an element of ac_0, then $[f, g]$ is convergent, as proved immediately above and determines a full and hence a normed linear functional on d^1 with $\|\Phi\|=\|g\|$. We state our result as

THEOREM III. $(d^1)^*=(d^1)'=ac_0$.

5. q^1, A NON-α-REFLEXIVE SPACE

We now introduce the linear space q^1 which is the species of all complex sequences $f=(f(n))_{n=1}^{\infty}$ for which a strictly increasing sequence of natural numbers $N(1), N(2), N(3), \ldots$ can be indicated such that $\sum_{i=0}^{\infty} |\sum_{k=n(i)+1}^{n(i+1)} f(k)|$ is bounded, whenever $0 \leqslant n(0) < N(1)$, and $N(i) \leqslant n(i) < N(i+1)$ for each $i \geqslant 1$. Classically speaking, q^1 coincides with w^1, but, in intuitionism, it contains w^1 without coinciding with it, but is however congruent with it. Obviously q^1 is contained in d^1: the quasi-norm of d^1 will also be used in q^1. The closed unit sphere of q^1 is not complete with respect to the distance delimitation ω used in d^1. For, if the closed unit sphere of q^1 were complete with respect to ω, then as it contains the generating catalogue of the closed unit sphere of d^1 (viz., the species of all vectors with only finitely many rational components, and lying in the closed unit sphere of d^1), it must have coincided with the closed unit sphere of d^1, which is known to be contradictory. The cataloguing of the closed unit sphere of q^1 has been not determined here, but we determine the dual space of q^1 without using the cataloguing of the closed unit sphere of q^1 as in the previous cases.

Let Φ be a full linear functional defined on q^1 and let $g(k)=\Phi(e_k)$. Now the dressed spread $\Lambda(d^1)$ constructed above is contained in the space q^1, so by the same reasoning as given above we can prove that to the full linear functional Φ of q^1, a vector g of ac_0 can be assigned such that $\Phi(f)=[f, g]$.

If g is any vector of ac_0, then $[f, g]$ converges for every vector f of q^1 since it even lies in d^1; and this thus determines a full and hence normed linear functional of d^1, and hence, a fortiori, of q^1. We have proved

THEOREM IV. $(q^1)^* = (q^1)' = ac_0$.

It follows from this that $(q^1)^{\alpha} = (ac_0)^* = d^1$ so that q^1 cannot be α-reflexive. In fact, q^1 cannot possess any of the four types of reflexivity. It may be mentioned here that a theorem of Bishop ([15], ch. 9, theo. 10) is tantamount to the fact that normed catalogued linear spaces are α-reflexive. However, for all other dual pairs of linear spaces so far introduced in intuitionism, α-reflexivity is still found to hold even if neither space of the pair is normed, giving rise to a possible conjecture that Bishop's theorem might be of a wider scope and that α-reflexivity might hold even for all linear locally convex spaces. The remarkable example of the linear space q^1 refutes this.

6. THE DUAL SPACE OF ac

We now wish to determine the dual space of ac. Let Φ be a full linear functional defined on ac and let $\Phi(e_k)=g(k)$. Any vector f of ac can be expressed as $f=(f-f(\infty)e)+f(\infty)e$ in which $f-f(\infty)e=(f(n)-f(\infty))_{n=1}^{\infty}$ converges to zero, so that it lies in ac_0 and we have $\Phi(f)=\Phi(f-f(\infty)e)+$ $+f(\infty)\Phi(e)= \sum_{i=1}^{\infty} (f(i)-f(\infty))g(i)+f(\infty)g(0)$ (where $g(0)=\Phi(e)$. However, $f-f(\infty)e$ lies in ac_0, and so $(g(n))_{n=1}^{\infty}$ lies in d^1, i.e., $|\sum_{i=1}^{n} g(i)|$ is bounded. Thus to any full linear functional Φ there corresponds a vector $(g(n))_{n=0}^{\infty}$

such that $\sum_{i=1}^{\infty} g(i)$ is bounded and $\Phi(f)=f(\infty)g(0)+\sum_{i=1}^{\infty}(g(i)-f(\infty))g(i)$, The quasi-norm of the vectors in $(ac)^*$ has not been specified. However. if g generates a full linear functional Φ on ac which is bounded by 1 then we can say that g lies in the closed unit sphere of $(ac)^*$.

7. The space c and its dual

c is the normed space of all convergent sequences, the norm being defined by $\|f\|=\sup_n |f(n)|$, which supremum exists in view of the convergence of $f(1), f(2), \ldots$.

We now wish to determine the dual space of c. Let Φ be a full linear functional defined on c and let $\Phi(e_k)=g(k)$. Any vector f of c can be expressed as $f=(f-f(\infty)e)+f(\infty)e$ in which $f-f(\infty)e=(f(n)-f(\infty))_{n=1}^{\infty}$ converges to zero, so that it lies in c_0 and we have $\Phi(f)=\Phi(f-f(\infty)e)+$ $+f(\infty)\Phi(e)=\sum_{i=1}^{\infty}(f(i)-f(\infty))g(i)+f(\infty)g(0)$ (where $g(0)=\Phi(e)$). However, $f-f(\infty)e$ lies in c_0, and so $(g(n))_{n=1}^{\infty}$ lies in b^1, i.e., $\sum_{k=1}^{\infty}|g(k)|$ is bounded. Now here we define the quasi norm of g, $|||g|||=|g(0)-\sum_{k=1}^{\infty}g(k)|+$ $+\sum_{k=1}^{\infty}|g(k)|$, as the quasi-number core containing the bounded monotone sequence $(|g(0)-g(1)|+|g(1)|, |g(0)-\sum_{i=1}^{2}g(i)|+\sum_{i=1}^{2}|g(i)|, \ldots)$.

Now we introduce the space β^1 which is the species of all complex sequences $f=(f(n))_{n=0}^{\infty}$ with bounded $\sum_{i=1}^{\infty}|f(i)|$, the quasi-norm of any vector f being $|||f|||=|f(0)-\sum_{i=1}^{\infty}f(i)|+\sum_{i=1}^{\infty}|f(i)|$.

For each fixed k, for each fixed natural number n, we can choose exactly one assertion from those recognized as valid of the two assertions (i) $|g(0)-\sum_{i=2}^{n}g(i)|>2^{-k-1}$, (ii) $|g(0)-\sum_{i=1}^{n}g(i)|<2^{-k}$ and take $s_{n,k}(0)=$ $=\operatorname{sgn}\overline{(g(0)-\sum_{i=1}^{n}g(i))}$ if (i) has been chosen, but $s_{n,k}(0)=0$ if (ii) has been chosen. Also for each $i=1, 2, \ldots$, we can choose exactly one assertion from those recognized as valid of the two assertions (i) $|g(i)|>2^{-k-i-1}$, and (ii) $|g(i)|<2^{-k-i}$; taking $s_{n,k}(i)=\operatorname{sgn}\overline{g(i)}$ if (i) has been chosen, but $s_{n,k}(i)=0$ if (ii) has been chosen. Taking the vector

$$f_{n,k}=(s_{n,k}(1), \ldots, s_{n,k}(n), s_{n,k}(0), s_{n,k}(0), \ldots)$$

which belongs to the space c, we have

$$\Phi(f_{n,k})=s_{n,k}(0)g(0)+\sum_{i=1}^{n}(s_{n,k}(i)-s_{n,k}(0))g(i)=s_{n,k}(0)\ (g(0)-\sum_{i=1}^{n}g(i))+ + \sum_{i=1}^{n}s_{n,k}(i)\,g(i),$$

therefore,

$$(|g(0)-\sum_{i=1}^{n}g(i)|+\sum_{i=1}^{n}|g(i)|)-2^{-k}-2^{-k-1}-\ldots-2^{-k-n}$$

$$\not> (g(0)-\sum_{1}^{n}g(i))\,s_{n,k}(0)+\sum_{i=1}^{n}g(i)\,s_{n,k}(i)\not>\Phi(f_{n,k}),\ \text{i.e.,}$$

$$|g(0)-\sum_{i=1}^{n}g(i)|+\sum_{i=1}^{n}|g(i)|\not> C+2^{-k+1}$$

(where C is any bound of Φ on the closed unit sphere of β^1), for every n and arbitrary k so that $|g(0)-\sum_1^n g(i)|+\sum_1^n |g(i)| \not> C$ for every n. Thus we have proved that to each full linear functional Φ on c a vector g of β^1 can be assigned such that $\Phi(f)=[f-f(\infty)e, g]+f(\infty)g(0)$ and for every bounded monotone sequence $a_1, a_2, \ldots$ contained in $|||g|||$, and every n, a vector f can be indicated such that $\Phi(f) \not< a_n$.

Conversely, if f belongs to the closed unit sphere of c and g lies in β^1, then $f(\infty)\Phi(e)+\sum_{i=1}^{\infty}(f(i)-f(\infty))g(i)$ converges, as $(f(i)-f(\infty))_{i=1}^{\infty}$ lies in c_0 and g lies in b^1, therefore corresponding to given k, we can find an $N(k)$ such that $|\sum_{i=1}^{n+m}(f(i)-f(\infty))g(i)|<2^{-k}$ for $n \geqslant N(k)$, $m \geqslant 0$. However, for $n \geqslant N(k)$,

$$\Phi(f)=f(\infty)\,\Phi(e)+\sum_{i=1}^{n}(f(i)-f(\infty))\,g(i)+\sum_{i=n+1}^{\infty}(f(i)-f(\infty))\,g(i)$$

$$=f(\infty)(g(0)-\sum_{i=1}^{n} g(i))+\sum_{i=1}^{n} f(i)g(i)+\sum_{i=n+1}^{\infty}(f(i)-f(\infty)g(i) \not>$$

$$\not> |f(\infty)|\,|g(0)-\sum_{i=1}^{n} g(i)|+\sum_{i=1}^{n}|f(i)|\,|g(i)|+2^{-k} \not> \max(|f(1)|, |f(2)|, \ldots, |f(n)|,$$

$$|f(\infty)|)\,(|g(0)-\sum_{i=1}^{n} g(i)|+\sum_{i=1}^{n}|g(i)|)+2^{-k} \not> 1\cdot(|g(0)-$$

$$-\sum_{i=1}^{n} g(i)|+\sum_{i=1}^{n}|g(i)|)+2^{-k}.$$

Thus we have proved that every bound of $|||g|||$ is also a bound of Φ, which proves completely that $|||\Phi|||=|||g|||$. Thus we have proved

THEOREM V. $(c)^*=\beta^1$.

8. THE NORMED DUAL OF c

We now proceed to determine the normed dual space of c. Let Φ be a full normed linear functional defined on c. The vector $g=(g(n))_{n=0}^{\infty}$ given by $g(k)=\Phi(e_k)$, $g(0)=\Phi(e)$ lies in β^1 and $|g(0)-\sum_{i=1}^{n} g(i)|+\sum_{i=1}^{n}|g(i)| \not> \not> \|\Phi\|$ for every n as mentioned above.

Since $\|\Phi\|$ is the supremum of Φ, we conclude that for each k, a vector f_k belonging to the closed unit sphere of c can be indicated such that $\Phi(f_k)=f_k(\infty)g(0)+[f_k-f_k(\infty)e, g]>\|\Phi\|-2^{-k-1}$ and we can also indicate an $N=N(k)$ such that $|[{}^nf_k-f_k(\infty)^ne, g]-[f_k-f_k(\infty)e, g]|<2^{-k-1}$ for $n \geqslant N(k)$. Thus we have

$$\|\Phi\|-2^{-k}<|[{}^nf_k-f_k(\infty)^ne, g]+f_k(\infty)g(0)]|=|\sum_{i=1}^{n}(f_k(i)-f_k(\infty))g(i)$$

$$+f_k(\infty)g(0)|=|f_k(\infty)(g(0)-\sum_{i=1}^{n} g(i))+\sum_{i=1}^{n} f_k(i)g(i)|$$

$$<\max(|f_k(\infty)|, |f_k(1)|, \ldots, |f_k(n)|)(|g(0)-\sum_{i=1}^{n} g(i)|+\sum_{i=1}^{n}|g(i)|)$$

$$<1(|g(0)-\sum_{i=1}^{n} g(i)|+\sum_{i=1}^{n}|g(i)|).$$

Hence we have $\|\Phi\| - 2^{-k} < (|g(0) - \sum_{i=1}^{n} g(i)| + \sum_{i=1}^{n} |g(i)|) \not> \|\Phi\|$, from which it follows that $\lim_{n\to\infty} (|g(0) - \sum_{i=1}^{n} g(i)| + \sum_{i=1}^{n} |g(i)|)$ exists and is equal to $\|\Phi\|$, i.e., $\|\Phi\| = \|g\|$.

We now introduce the space λ^1 which is the species of all complex sequences $f = (f(n))_{n=0}^{\infty}$ for which $\sum_{i=1}^{\infty} |f(i)|$ converges, the norm of any vector f being taken to be $\|f\| = |f(0) - \sum_{k=1}^{\infty} f(k)| + \sum_{k=1}^{\infty} |f(k)|$. Thus we have proved that to each full normed linear functional Φ on c a vector g of λ^1 can be assigned such that $\Phi(f) = [f - f(\infty)e, g] + f(\infty)g(0)$, and $\|\Phi\| = \|g\|$.

Conversely, if g lies in λ^1, then it also lies in β^1 and determines a full linear functional Φ on c. We now prove that it possesses a supremum $\|g\|$ on the closed unit sphere of c. Taking the vector $f_{n,k}$ belonging to the closed unit sphere c taken in § 7, we have $|g(0) - \sum_{i=1}^{n} g(i)| + \sum_{i=1}^{n} |g(i)| - 2^{-k+1} \not> \Phi(f_{n,k})$ as shown above. Therefore, $\Phi(f_{n,k}) \not< \|g\| - 2^{-k+2}$ for a sufficiently large n. We state this result as

THEOREM VI. $(c)' = \lambda^1$.

9. THE DUAL OF β^1

On the closed unit sphere of β^1, we define the same distance delimitation ω as on the closed unit sphere of b^1. We shall prove that the closed unit sphere of β^1 is complete with respect to ω. Let $f^1, f^2, f^3, \ldots$ be a fundamental sequence of elements of the closed unit sphere of β^1. However, corresponding to given n we can indicate an $N(n)$ such that n belongs to $\omega(f^N, f^{N+m})$, i.e., $|f^N(i) - f^{N+m}(i)| < 2^{-n}$ for $0 \leqslant i \leqslant n$. It follows from this, that for each fixed $i (0 \leqslant i \leqslant n)$, $f^1(i), f^2(i), \ldots$ is a convergent sequence of complex numbers: let its limit be $f(i)$. We shall show that f belongs to the closed unit sphere of β^1. We know that $|f^m(0) - \sum_{i=1}^{n} f^m(i)| + \sum_{i=1}^{n} |f^m(i)| \not> 1$ for every $n \geqslant 1$ and $m \geqslant 1$. However, for each fixed n, and each given positive number k an $m_0 = m_0(k, n)$ can be indicated such that $|f(i) - f^m(i)| < 2^{-k}/(2n+1)$ for each $i = 0, 1, \ldots n$, whenever $m \geqslant m_0$. Hence $|f(i)| \not> |f^m(i)| + 2^{-k}/(2n+1)$ for $0 \leqslant i \leqslant n$, so that $\sum_{i=1}^{n} |f(i)| \not> \sum_{i=1}^{n} |f^m(i)| + n \cdot 2^{-k}/(2n+1)$ and

$$
\begin{aligned}
|f(0) - \sum_{i=1}^{n} f(i)| &\not> |f(0) - f^m(0)| + |f^m(0) - \sum_{i=1}^{n} f^m(i)| + |\sum_{i=1}^{n} f^m(i) - \sum_{i=1}^{n} f(i)| \\
&\not> 2^{-k}/(2n+1) + |f^m(0) - \sum_{i=1}^{n} f^m(i)| + |\sum_{i=1}^{n} f^m(i) - \sum_{i=1}^{n} f(i)| \\
&\not> 2^{-k}/(2n+1) + |f^m(0) - \sum_{i=1}^{n} f^m(i)| + n \cdot 2^{-k}/(2n+1).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
|f(0) - \sum_{i=1}^{n} f(i)| + \sum_{i=1}^{n} |f(i)| &\not> \sum_{i=1}^{n} |f^m(i)| + n \cdot 2^{-k}/(2n+1) + |f^m(0) - \\
- \sum_{i=1}^{n} f^m(i)| + 2^{-k}/(2n+1) + n \cdot 2^{-k}/(2n+1) &\not> |f^m(0) - \sum_{i=1}^{n} f^m(i)| + \sum_{i=1}^{n} |f^m(i)| + 2^{-k} \not> 1 + 2^{-k}.
\end{aligned}
$$

As k is arbitrary, $|f(0)-\sum_{i=1}^{n} f(i)|+\sum_{i=1}^{n}|f(i)| \ngtr 1$ for every n. Hence f lies in the closed unit sphere of β^1, i.e., the closed unit sphere of β^1 is complete.

It can be shown exactly as in the case of b^1, that the cataloguing of the closed unit sphere of β^1 is of the first kind. As the cataloguing of the closed unit sphere of β^1 is of the first kind, every full linear functional of β^1 necessarily possesses a supremum, so that its normed dual space $(\beta^1)'$ is identical with its dual space $(\beta^1)^*$.

Now, given any full linear functional Φ of β^1 which possesses the supremum $\|\Phi\|$ on the closed unit sphere of β^1, we shall show that it determines a vector $g=(g(n))_{n=1}^{\infty}$ $(g(k)=\Phi(e_0+e_k)$ $k=1, 2, 3, \ldots)$ of c, i.e., the sequence $g(1), g(2), \ldots$ converges.

We dress the fan L (whose direction consists of two nodes of order one, and in which each node containing only 1's has exactly two immediate descendants, while each node in which a 2 occurs somewhere has exactly one immediate descendant; cf. [13], § 1), by replacing each node $a_1, a_2, \ldots, a_n$ by the dressed node 1, $(a_2-1), a_3-1), \ldots, (a_n-1)$ to obtain the dressed fan direction $L(\beta^1)$. Obviously, $L(\beta^1)$ is contained in the closed unit sphere of β^1. Since Φ is full on $L(\beta^1)$, it follows that to each k, an $N(k)$ can be assigned such that $|\Phi(f)-\Phi(f')|<2^{-k}$ for any f and f' in $L(\beta^1)$ passing through the dressed node consisting of $(1, 0, 0, \ldots, 0)$ (N zeros). Applying this to the vector (e_0+e_n) and (e_0+e_{n+m}) for $n>N(k)$ and $m\geqslant 0$, we have $|\Phi(e_0+e_n)-\Phi(e_0+e_{n+m})|<2^{-k}$, i.e., $|g(n)-g(n+m)|<2^{-k}$ for $n>N(k)$ and $m\geqslant 0$, from which follows that g lies in c. However, $|g(n)|=|\Phi(e_0+e_n)| \ngtr \ngtr \|\Phi\|\cdot 1$ for every n, so $\sup_n |g(n)| \ngtr \|\Phi\|$, i.e., $\|g\| \ngtr \|\Phi\|$. Thus we have proved that to each full linear functional Φ on β^1 a vector g of c can be assigned such that $\Phi(f)=f(0)g(\infty)+\sum_{i=1}^{\infty} f(i)(g(i)-g(\infty))$ and $\|g\| \ngtr \|\Phi\|$.

Conversely, let f be a vector of the closed unit sphere of β^1 and let g belong to c. The series $f(0)g(\infty)+\sum_{i=1}^{\infty} f(i)(g(i)-g(\infty))$ is obviously convergent. Hence Φ defined by $\Phi(f)=f(0)g(\infty)+\sum_{i=1}^{\infty} f(i)(g(i)-g(\infty))$ for each f in β^1 is a linear functional on β^1. However, taking a sufficiently large n, we have

$$|\Phi(f)|=|f(0)g(\infty)+\sum_{i=1}^{n} f(i)(g(i)-g(\infty))+\sum_{i=n+1}^{\infty} f(i)\ (g(i)-g(\infty))|$$

$$\ngtr |f(0)g(\infty)+\sum_{i=1}^{n} f(i)(g(i)-g(\infty))|+2^{-k}$$

$$\ngtr |g(\infty)(f(0)-\sum_{i=1}^{n} f(i))+\sum_{i=1}^{n} f(i)g(i)|+2^{-k}$$

$$\ngtr \max\,(|g(1)|, |g(2)|, \ldots, |g(n)|, |g(\infty)|)\,(|f(0)-\sum_{i=1}^{n} f(i)|+\sum_{i=1}^{n} |f(i)|) \ngtr \|g\|\cdot 1,$$

i.e., $\|\Phi\| \ngtr \|g\|$, so that $\|\Phi\|=\|g\|$, with which we have completely proved

THEOREM VII. $(\beta^1)^*=(\beta^1)'=c$.

10. The dual of λ^1

Let Φ be a full linear functional defined on λ^1 and let $\Phi(e_0)=g(0)$: $\Phi(e_0+e_k)=g(k)$ for $k=1, 2, 3, \ldots$. If in the space b^∞ we take the quasi-norm of $f=f(n))_{n=0}^\infty$ as the quasi-number core containing the bounded monotone sequence $(|f(0)|, \max(|f(0)|, |f(1)|), \ldots)$, then it can be proved that $(\lambda^1)^*=b^\infty$ in the same way as $(l^1)^*=b^\infty$. It can also be proved that $(\lambda^1)'=l^\infty$. In determining the second dual space $(\lambda^1)^{**}$, if we apply the representation $\Phi(f^*)=f^*(f)=\sum_{i=1}^\infty f(i)f^*(i)+(f(0)-\sum_{i=1}^\infty f(i))f^*(0)$, then we can prove that $(\lambda^1)^{**}=\lambda^1$ and $(\lambda^1)'^*=(\lambda^1)''=\lambda^1$.

I wish to express my most sincere thanks to Dr. Ashvinikumar for his valuable guidance and critical supervision throughout the preparation of this paper. I also wish to express my sincere gratitude to Prof. A. Heyting, who always shows a keen interest in my mathematical activity, for critically examining this article.

Department of Mathematics
Indian Institute of Technology
Kanpur, U.P., India

BIBLIOGRAPHY

1. Brouwer, L. E. J., Points and spaces. Can. J. Math. 6, 1–17 (1954).
2. Heyting, A., Intuitionism, an introduction. Amsterdam 1966.
3. ———, Note on the Riesz-Fischer theorem. Proc. Akad. Amsterdam 54, 35–40 (1951).
4. ———, Espaces de Hilbert et intuitionnisme. Colloque internationaux du C.N.R.S. Paris 1953, 59–63.
5. ———, Intuïtionistische theorie der Hilbert-ruimten. (Unpublished lecture notes in Netherlandish. Amsterdam (1949–50)).
6. Ashvinikumar, Hilbert Spaces in intuitionism-Hilbertaj spacoj en intuiciismo. Thesis. Amsterdam, 1966.
7. ———, Über katalogisierte Räume. Comp. Math. 21, 431–456 (1960).
8. ———, On the intuitionist theory of Stieltjes integration and its applications. Proc. Kon. Ned. Ak. v. Wet. Ser. A, 73, 62–76 (1970).
9. ———, On Brouwer-Stieltjes Integration. Proc. Kon. Ned. Ak. v. Wet. Ser. A, 73, 161–171 (1970).
10. ———, A note on quasi-numbers. (To appear).
11. ——— and S. L. Shukla, Intuitionist determination of dual spaces of certain catalogued linear spaces. I, II. Proc. Kon. Ned. Ak. v. Wet. Ser. A, 74, 3 (1971).
12. Shukla, S. L., On a new method in intuitionist linear analysis. (To appear).
13. ———, On intuitionist analogues of classically inseparable spaces. (To appear).
14. Hill, J. D., On the space (γ) of convergent series, Tôhoku Math. J. 45, 332–337 (1939).
15. Hardy, G. H., Divergent Series, Oxford, 1949.
16. Bishop, E., Foundations of constructive analysis. New York 1967.
17. Köthe, G., Topologische lineare Räume I., Berlin-Göttingen-Heidelberg 1960.
18. Kantorovich, L. V. and G. P. Akilov, Functional analysis in normed spaces. Oxford-London-New York etc. 1964. (Translated from the Russian "Funktsional'nyi analiz v normirovannykh prostranstvakh", Moscow 1959.)
19. Riesz, F. and B. Sz.-Nagy, Leçons d'analyse fonctionelle. Budapest 1953.